REMARKS ON THE ENERGY PRINCIPLE IN MAGNETOHYDRODYNAMICS

(ZAMECHANIIA K ENERGETICHESKOMU PRINTSIPU V MAGNITNOI GIDRODINAMIKE)

PMM Vol.25, No.4, 1961, pp. 760-764

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(Received January 28, 1961)

The energy principle, obtained in the papers by Lundquist [1,2], Bernstein and others [3], has found broad application in the solution of problems of stability of equilibrium magnetohydrodynamic configurations. The energy principle is analogous to known theorems [4,5,6] on the stability of the equilibrium condition of a system of material points. The proof of the energy principle, carried out in [3], is based upon the series expansion in small displacements of an ideal conducting fluid along a complete system of normal vibrations.

In this paper we shall produce proofs of the stability theorems of equilibrium configurations of an ideal conducting fluid by the use of the Liapunov function.

Note that the totality of solutions of the Cauchy equations of motion for small displacements of a nonviscous ideally conducting fluid from the condition of equilibrium determines a general system. Zubov has used [6] the analog of Liapunov functions in the proof of theorems of stability of invariant ensembles of a general system. Movchan [7] applied Liapunov functions in the investigation of elastic systems.

1. As shown in [3,8,9], the equations of motion for small displacements $\xi(\mathbf{r},\ t)$ of a nonviscous ideally conducting fluid from the equilibrium condition have the form

$$\rho\ddot{\xi}_i = F_i(\xi) \qquad (i = 1, 2, 3) \tag{1}$$

where a point above a symbol indicates differentiation with respect to time and F is a linear self-adjoint operator.

$$\mathbf{F}(\xi) = \nabla(\xi\cdot\nabla p) + \gamma\nabla(p\,\mathrm{div}\,\xi) + \frac{1}{4\pi}(\mathrm{rot\,rot}\,(\xi\times\mathbf{H})\times\mathbf{H}) + \frac{1}{4\pi}\mathrm{rot}\,\mathbf{H}\times\mathrm{rot}\,(\xi\times\mathbf{H})$$

where ρ, p and $\mathbf{H}$ are equilibrium values of density, pressures of fluid and magnetic field, γ is the adiabatic exponent.

We shall, for simplicity, assume that the fluid occupies a finite volume V, bounded by the surface S, where density ρ and displacements ξ become zero on S.

Equation (1) has an energy integral

$$E = T + U = \text{const} \tag{2}$$

where

$$T\{\dot{\xi}\} = \frac{1}{2}\int_V \rho\dot{\xi}^2\,d\mathbf{r}, \qquad U\{\xi\} = -\frac{1}{2}\int_V \xi\,\mathbf{F}(\xi)\,d\mathbf{r} \tag{3}$$

We shall introduce notations

$$\|\xi\| = \sup\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}, \qquad \|\dot{\xi}\| = \sup\sqrt{\dot{\xi}_1^2 + \dot{\xi}_2^2 + \dot{\xi}_3^2}$$

It is clear that

(1) $T = U = 0$ for $\|\xi\| = 0$, $\|\dot{\xi}\| = 0$

(2) $T \to 0$, $U \to 0$ for $\|\xi\| \to 0$, $\|\dot{\xi}\| \to 0$

(3) $T\{\dot{\xi}\} \geqslant 0$

We shall assume $\xi(\mathbf{r})$ to be twice continuously differentiable functions of $\mathbf{r} \in V$. Let $\xi = \xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))$ be the solution of Equation (1), satisfying the initial conditions

$$\xi = \xi_0(\mathbf{r}), \quad \dot{\xi} = \dot{\xi}_0(\mathbf{r}) \quad \text{for } t = 0$$

We shall now give the determination of stability, asymptotic stability and instability of equilibrium conditions of a given system.

The condition of equilibrium $\xi = 0$, $\dot{\xi} = 0$ is stable, if for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ may be found such $\delta_1 > 0$ and $\delta_2 > 0$, that if $\|\xi_0(\mathbf{r})\| < \delta_1$ and $\|\dot{\xi}_0(\mathbf{r})\| < \delta_2$, then

$$\|\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\| < \varepsilon_1, \quad \|\dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\| < \varepsilon_2 \quad \text{for } t \geqslant 0$$

The condition of equilibrium is asymptotically stable, if it is stable, and also

$$\|\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\| \to 0, \quad \|\dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\| \to 0 \quad \text{for } t \to +\infty.$$

The condition of equilibrium is not stable if there exist at least one $\epsilon_1 > 0$ and $\epsilon_2 > 0$ such that for any $\delta_1 > 0$ and $\delta_2 > 0$ there always exist such $\xi_0(\mathbf{r})$ and $\dot{\xi}_0(\mathbf{r})$, $\|\xi_0(\mathbf{r})\| < \delta_1$, $\|\dot{\xi}_0(\mathbf{r})\| < \delta_2$ that at least one of the following inequalities will be satisfied:

$$\|\xi(t,\mathbf{r},\xi_0(\mathbf{r}),\dot{\xi}_0(\mathbf{r}))\| \geqslant \varepsilon_1, \qquad \|\dot{\xi}(t,\mathbf{r},\xi_0(\mathbf{r}),\dot{\xi}_0(\mathbf{r}))\| \geqslant \varepsilon_2$$

at least for one of the values $t \geqslant 0$.

2. We shall now formulate and prove the theorem of the stability of equilibrium conditions of an ideally conductive nonviscous fluid.

Theorem 1. (Necessary condition of stability.) In order that the condition of equilibrium $\xi = 0$, $\dot{\xi} = 0$ be stable, it is necessary that $U\{\xi\} \geqslant 0$.

Proof. Let a given condition of equilibrium be stable. We shall show that in that case $U\{\xi\} \geqslant 0$. We shall assume that there exists a $\xi^*(\mathbf{r})$, such that

$$U\{\xi^*(\mathbf{r})\} < 0, \qquad \|\xi^*(\mathbf{r})\| \neq 0$$

Assume that

$$V\{\xi, \dot{\xi}\} = \int_V \rho\, \xi\dot{\xi}\, d\mathbf{r} \tag{4}$$

If ξ is the solution of (1), then

$$\frac{dV\{\xi, \dot{\xi}\}}{dt} = 2\left(T\{\dot{\xi}\} - U\{\xi\}\right)$$

If for any given $\delta_1^* > 0$ and $\delta_2^* > 0$, there always exist $\xi_0^*(\mathbf{r})$ and $\dot{\xi}_1^*(\mathbf{r})$ such that

$$0 < \|\xi_0^*(\mathbf{r})\| < \delta_1^*, \qquad 0 < \|\dot{\xi}_0^*(\mathbf{r})\| < \delta_2^*$$

$$U\{\xi_0^*\} < 0, \qquad V\{\xi_0^*, \dot{\xi}_0^*\} > 0, \qquad T\{\dot{\xi}_0^*\} + U\{\xi_0^*\} < 0$$

then for solutions of Equation (1) with initial conditions

$$\xi = \xi_0^*(\mathbf{r}), \quad \dot{\xi} = \dot{\xi}_0^*(\mathbf{r}) \quad \text{for } t = 0$$

we shall have

$$\frac{dV\{\xi, \dot{\xi}\}}{dt} \geqslant 2\mu > 0 \quad \text{for } t \geqslant 0 \quad (-\mu = T\{\dot{\xi}_0^*\} + U\{\xi_0^*\} < 0)$$

Hence it follows that

$$V \geqslant V_0 + \mu t \to +\infty \quad \text{for } t \to +\infty, \qquad (V_0 = V \text{ for } t = 0) \tag{5}$$

Since the condition of equilibrium is stable, then for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$ there are found $\delta_1 > 0$ and $\delta_2 > 0$ such that if $\|\xi_0(\mathbf{r})\| < \delta_1$ and $\|\dot{\xi}_0(\mathbf{r})\| < \delta_2$, then

$$\|\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\| < \varepsilon_1, \qquad \|\dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\| < \varepsilon_2 \quad \text{for } t \geqslant 0$$

We shall now choose $\delta_1{}^*$ and $\delta_2{}^*$ corresponding to δ_1 and δ_2. Then for $t \geqslant 0$ we find that

$$\|\xi(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r}))\| < \varepsilon_1, \qquad \|\dot{\xi}(t, \mathbf{r}, \xi_0{}^*(\mathbf{r}), \dot{\xi}_0{}^*(\mathbf{r}))\| < \varepsilon_2$$

Hence if follows that for the solutions under consideration

$$|V| \leqslant \varepsilon_1 \varepsilon_2 M \qquad \left(M = \int_V \rho d\mathbf{r}\right)$$

This contradicts (5). Consequently the assumption that $U\{\xi\} < 0$ is false. The theorem is proven.

From Theorem 1 it follows that if there exist such $\xi(\mathbf{r})$ that $U\{\xi(\mathbf{r})\} < 0$, then a given condition of equilibrium is not stable.

It is easily proven that the condition of equilibrium of an ideally conducting nonviscous fluid cannot be asymptotically stable.

3. If a given fluid is viscous, then the equations of motion will have the form

$$\rho \ddot{\xi}_i = F_i(\xi) + f_i(\dot{\xi}) \qquad (i = 1, 2, 3) \tag{6}$$

The force of viscous friction f_i equals

$$f_i = \frac{\partial}{\partial x_k}\left[\eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)\right] + \frac{\partial}{\partial x_i}\left(\zeta \frac{\partial v_l}{\partial x_l}\right) \tag{7}$$

where η and ζ are first and second coefficients of viscosity, $v_i = \dot{\xi}_i$.

From (6) it is easily found that

$$d(T + U)/dt = -W \leqslant 0 \tag{8}$$

where

$$W\{\dot{\xi}\} = -\int_V v_i f_i \, d\mathbf{r} = \frac{1}{2}\int_V \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_l}{\partial x_l}\right)^2 d\mathbf{r} + \int_V \zeta\left(\frac{\partial v_l}{\partial x_l}\right)^2 d\mathbf{r} \tag{9}$$

We shall now investigate the influence of viscosity forces on the stability of condition of equilibrium. In the case of an incompressible fluid this problem was considered by Hare, using another method [10].

Theorem 2. If there exist $\xi(\mathbf{r})$ such that $U\{\xi(\mathbf{r})\} < 0$, then the condition of equilibrium is not stable even in the presence of viscous forces.

Proof. We shall assume that the condition of equilibrium is stable.

Choose $\epsilon_1 > 0$ and $\epsilon_2 > 0$; then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that if

$$\|\xi_0(\mathbf{r})\| < \delta_1, \qquad \|\dot{\xi}_0(\mathbf{r})\| < \delta_2 \tag{10}$$

then for $t \geqslant 0$

$$\|\xi(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r})\| < \varepsilon_1, \qquad \|\dot{\xi}(t, \mathbf{r}, \xi_0(\mathbf{r}), \dot{\xi}_0(\mathbf{r}))\| < \varepsilon_2 \tag{11}$$

Assume

$$V\{\xi, \dot{\xi}\} = \int_V \rho\xi\dot{\xi}\, d\mathbf{r} + \frac{1}{4}\int_V \eta\left(\frac{\partial\xi_i}{\partial x_k} + \frac{\partial\xi_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial\xi_l}{\partial x_l}\right)^2 d\mathbf{r} + \frac{1}{2}\int_V \zeta\left(\frac{\partial\xi_l}{\partial x_l}\right)^2 d\mathbf{r} \tag{12}$$

If ξ is the solution of Equation (6), then

$$dV / dt = 2(T - U) \tag{13}$$

There exist $\xi_0^*(\mathbf{r})$ and $\dot{\xi}_0^*(\mathbf{r})$ such that

$$\|\xi_0^*(\mathbf{r})\| < \delta_1, \qquad \|\dot{\xi}_0^*(\mathbf{r})\| < \delta_2$$
$$\mu = -T\{\dot{\xi}_0^*(\mathbf{r})\} - U\{\xi_0^*(\mathbf{r})\} > 0, \; V\{\xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r})\} > 0 \tag{14}$$

Since for $t \geqslant 0$

$$\|\xi(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r}))\| < \varepsilon_1 \qquad \|\dot{\xi}(t, \mathbf{r}, (\xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r}))\| < \varepsilon_2$$

then there exist $\lambda > 0$ such that for $t \geqslant 0$

$$V\{\xi(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r})), \quad \|\dot{\xi}(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r}))\} < \lambda \tag{15}$$

On the other hand, since for $t \geqslant 0$

$$T\{\dot{\xi}(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r}))\} + U\{\xi(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r}))\} \leqslant -\mu < 0$$

we find that for $t \geqslant 0$

$$\frac{d}{dt}V\{\xi(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r})), \qquad \dot{\xi}(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r}))\} \geqslant 2\mu > 0 \tag{16}$$

Consequently, for $t \to +\infty$

$$V\{\xi(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r})), \qquad \dot{\xi}(t, \mathbf{r}, \xi_0^*(\mathbf{r}), \dot{\xi}_0^*(\mathbf{r}))\} \to +\infty \tag{17}$$

which contradicts (15). The contradiction arrived at shows that the assumption made with respect to the stability of the condition of equilibrium is false. The theorem is proven.

Theorem 2 is analogous to the known Kelvin [5] theorem on the influence of dissipative forces on the stability of the condition of equilibrium of a system of material points.

Note. The use of functions (4) and (12) in the proofs of Theorems 1 and 2 should be analogous to using the Liapunov function, in its quadratic form, as was done by Chetaev [5, 11] in the proofs of some theorems on stability.

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Translated by J.R.W.